

Towards the classification of integrable differential-difference equations in $2 + 1$ dimensions

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Abstract

We address the problem of classification of integrable differential-difference equations in $2+1$ dimensions with one/two discrete variables. Our approach is based on the method of hydrodynamic reductions and its generalisation to dispersive equations as proposed in [10, 11]. We obtain a number of classification results of scalar integrable equations including that of the intermediate long wave and Toda type.

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1 Introduction

This paper is aimed at the classification of scalar $2 + 1$ dimensional integrable equations of the general form

$$u_t = F(u, w), \quad (1)$$

where $u(x, y, t)$ is a scalar field, $w(x, y, t)$ is the nonlocal variable, and F is a differential/difference operator in the independent variables x and y (the explicit form of w and F will be specified in what follows). Publications [10, 11] provide a novel perturbative approach to the classification of integrable equations of the form (1) which have non-degenerate dispersionless limit. This approach is based on the requirement that all hydrodynamic reductions [9] of the dispersionless limit can be deformed into reductions of the full dispersive equation (a brief description of this method is included in Sect. 2). For the simplest nonlocality $w = D_x^{-1} D_y u$ (or, equivalently, $w_x = u_y$), the paper [11] gives a complete list of integrable third order PDEs of the form

$$u_t = \varphi u_x + \psi u_y + \eta w_y + \epsilon(\dots) + \epsilon^2(\dots), \quad (2)$$

where φ, ψ and η are functions of u and w , while the terms at ϵ and ϵ^2 are assumed to be homogeneous differential polynomials of the order two and three in the x - and y -derivatives of u and w .

Theorem 1 [11] *Up to invertible transformations, the examples below provide a complete list of integrable third order equations (2) with non-degenerate dispersionless limit:*

<i>KP equation</i>	$u_t = uu_x + w_y + \epsilon^2 u_{xxx},$
<i>modified KP equation</i>	$u_t = (w - \frac{u^2}{2})u_x + w_y + \epsilon^2 u_{xxx},$
<i>Gardner equation</i>	$u_t = (\beta w - \frac{\beta^2}{2} u^2 + \delta u)u_x + w_y + \epsilon^2 u_{xxx},$
<i>VN equation</i>	$u_t = (uw)_y + \epsilon^2 u_{yyy},$
<i>modified VN equation</i>	$u_t = (uw)_y + \epsilon^2 \left(u_{yy} - \frac{3}{4} \frac{u_y^2}{u} \right)_y,$
<i>HD equation</i>	$u_t = -2wu_y + uw_y - \frac{\epsilon^2}{u} \left(\frac{1}{u} \right)_{xxx},$
<i>deformed HD equation</i>	$u_t = \frac{\delta}{u^3} u_x - 2wu_y + uw_y - \frac{\epsilon^2}{u} \left(\frac{1}{u} \right)_{xxx},$
<i>Equation E_1</i>	$u_t = (\beta w + \beta^2 u^2)u_x - 3\beta u u_y + w_y + \epsilon^2 [B^3(u) - \beta u_x B^2(u)],$
<i>Equation E_2</i>	$u_t = \frac{4}{3} \beta^2 u^3 u_x + (w - 3\beta u^2)u_y + uw_y + \epsilon^2 [B^3(u) - \beta u_x B^2(u)],$

here $B = \beta u D_x - D_y$, $\beta, \delta = \text{const.}$

Although most of these examples are well-known [17, 31], the last three equations are apparently new. We refer to [11] for technical details, Lax pairs, etc. Based on the same method, in [24] the above classification was extended to fifth order scalar equations, while the paper [16] provides a complete list of integrable two-component systems of Davey-Stewartson type (all with the same nonlocality $w_x = u_y$). See also [1, 19, 26] and references therein for the existing classification results of integrable systems in $2+1$ D.

Remark. The search for integrable equations (2) with ‘nested’ non-localities of the form $w_x = u_y$, $v_x = f(u, w)_y$ leads to commuting flows of equations from Theorem 1, and does not give essentially new examples.

The aim of this work is to extend the above classification to various differential-difference non-localities w such as

$$\Delta_x^+ w = \frac{T_x + 1}{2} u_y, \quad w_x = \Delta_y^+ u, \quad \Delta_x^+ w = \Delta_y^+ u.$$

The first two cases are known as the intermediate long wave and the Toda type non-localities, respectively. We use the following standard notation for the ϵ -shift operators and the forward/backward discrete derivatives,

$$T_x f(x, y) = f(x + \epsilon, y), \quad T_x^{-1} f(x, y) = f(x - \epsilon, y), \quad \Delta_x^+ = \frac{T_x - 1}{\epsilon}, \quad \Delta_x^- = \frac{1 - T_x^{-1}}{\epsilon},$$

same for T_y , T_y^{-1} , Δ_y^+ , Δ_y^- . Note that all our non-localities reduce to $w_x = u_y$ in the dispersionless limit $\epsilon \rightarrow 0$.

Here is a brief summary of our classification results (see Sect. 3 for the corresponding Lax pairs, dispersionless limits, etc). In Sect. 3.1 we consider two classes of equations with the nonlocality of intermediate long wave type. Our results are summarised in Theorems 2 and 3 below:

Theorem 2 *The following examples constitute a complete list of integrable equations of the form*

$$u_t = \varphi u_x + \psi u_y + \tau w_x + \eta w_y + \epsilon(\dots) + \epsilon^2(\dots),$$

where w is the non-locality of intermediate long wave type, $\Delta_x^+ w = \frac{T_x + 1}{2} u_y$, the coefficients $\varphi, \psi, \tau, \eta$ are functions of u, w , and the terms at ϵ and ϵ^2 are assumed to be homogeneous differential polynomials of the order two and three in the x - and y -derivatives of u and w :

$$\begin{aligned} u_t &= uu_y + w_y, \\ u_t &= (w + \alpha e^u) u_y + w_y, \\ u_t &= u^2 u_y + (uw)_y + \frac{\epsilon^2}{12} u_{yyy}, \\ u_t &= u^2 u_y + (uw)_y + \frac{\epsilon^2}{12} \left(u_{yy} - \frac{3}{4} \frac{u_y^2}{u} \right)_y. \end{aligned}$$

The first example appeared in [7] as a differential-difference analogue of the KP equation, see also [27]. It can be viewed as a $2 + 1$ dimensional integrable version of the intermediate long wave equation [35]. The last two examples are differential-difference versions of the Veselov-Novikov and the modified Veselov-Novikov equations, respectively (the third example appeared previously in [25]). The second and the fourth equations seem to be new.

Theorem 3 *The following examples constitute a complete list of integrable equations of the form*

$$u_t = \psi u_y + \eta w_y + f \Delta_x^+ g + p \Delta_x^- q,$$

where w is the non-locality of intermediate long wave type, $\Delta_x^+ w = \frac{T_x+1}{2}u_y$, and ψ, η, f, g, p, q are functions of u and w :

$$\begin{aligned} u_t &= uu_y + w_y, \\ u_t &= (w + \alpha e^u)u_y + w_y, \\ u_t &= wu_y + w_y + \frac{\Delta_x^+ + \Delta_x^-}{2}e^{2u}, \\ u_t &= wu_y + w_y + e^u(\Delta_x^+ + \Delta_x^-)e^u. \end{aligned}$$

Here the first two examples are the same as in Theorem 2, while the third appeared in [18]. The fourth equation is apparently new.

In Sect. 3.2 we consider the case of integrable equations with the Toda non-locality. The main result is as follows:

Theorem 4 *The following examples constitute a complete list of integrable equations of the form*

$$u_t = \varphi u_x + f \Delta_y^+ g + p \Delta_y^- q,$$

where w is the non-locality of Toda type, $w_x = \Delta_y^+ u$, and φ, f, g, p, q are functions of u and w :

$$\begin{aligned} u_t &= u \Delta_y^- w, \\ u_t &= (\alpha u + \beta) \Delta_y^- e^w, \\ u_t &= e^w \sqrt{u} \Delta_y^+ \sqrt{u} + \sqrt{u} \Delta_y^- (e^w \sqrt{u}), \end{aligned}$$

here $\alpha, \beta = \text{const.}$

The first example is the well-known Toda equation, while the second is equivalent to the Volterra (Toda) chain when $\alpha \neq 0$ ($\alpha = 0$), respectively. The third equation seems to be new.

In Sect. 3.3 we discuss the case of integrable equations with two discrete variables. Our main result is

Theorem 5 *The following examples constitute a complete list of integrable equations of the form*

$$u_t = f \Delta_x^+ g + h \Delta_x^- k + p \Delta_y^+ q + r \Delta_y^- s,$$

where w is the fully discrete non-locality of the form $\Delta_x^+ w = \Delta_y^+ u$, and f, g, h, k, p, q, r, s are functions of u and w :

$$\begin{aligned} u_t &= u \Delta_y^- (u - w), \\ u_t &= u (\Delta_x^+ + \Delta_y^-) w, \\ u_t &= (\alpha e^{-u} + \beta) \Delta_y^- e^{u-w}, \\ u_t &= (\alpha e^u + \beta) (\Delta_x^+ + \Delta_y^-) e^w, \\ u_t &= \sqrt{\alpha - \beta e^{2u}} \left(e^{w-u} \Delta_y^+ \sqrt{\alpha - \beta e^{2u}} + \Delta_y^- (e^{w-u} \sqrt{\alpha - \beta e^{2u}}) \right), \end{aligned}$$

here $\alpha, \beta = \text{const.}$

In equivalent form, the last example is known as the 2 + 1 dimensional analogue of the modified Volterra lattice [30].

Although fully discrete 3D equations are outside the scope of this paper, they can be dealt with in a similar way, suggesting an alternative to the standard approach based on the multidimensional consistency [2, 3, 23, 29]. The only constraint required by our method is that of the non-degeneracy of the dispersionless limit (see Sect. 2). Numerous examples of this type are provided by various equivalent forms of the Hirota equation [15, 22] governing KP/BKP/Toda hierarchies. These include the following difference equations (we prefer to write them using discrete derivatives Δ rather than shifts T : in this form the dispersionless limit becomes more easily seen):

$$(\Delta_x^+ u - \Delta_y^+ u) \Delta_x^+ \Delta_y^+ u + (\Delta_t^+ u - \Delta_x^+ u) \Delta_x^+ \Delta_t^+ u + (\Delta_y^+ u - \Delta_t^+ u) \Delta_y^+ \Delta_t^+ u = 0,$$

$$\Delta_x^+ \left(\ln \frac{\Delta_t^+ u}{\Delta_y^+ u} \right) + \Delta_y^+ \left(\ln \frac{\Delta_x^+ u}{\Delta_t^+ u} \right) + \Delta_t^+ \left(\ln \frac{\Delta_y^+ u}{\Delta_x^+ u} \right) = 0,$$

$$\alpha e^{\Delta_x^+ \Delta_y^+ u} + \beta e^{\Delta_x^+ \Delta_t^+ u} + \gamma e^{\Delta_y^+ \Delta_t^+ u} = 0,$$

$$\alpha e^{\Delta_x^+ \Delta_x^- u} + \beta e^{\Delta_y^+ \Delta_y^- u} + \gamma e^{\Delta_t^+ \Delta_t^- u} = 0,$$

etc, here α, β, γ are arbitrary constants. The corresponding dispersionless limits $\epsilon \rightarrow 0$ can be obtained by replacing discrete derivatives by partial derivatives. This gives

$$(u_x - u_y)u_{xy} + (u_t - u_x)u_{xt} + (u_y - u_t)u_{yt} = 0,$$

$$\left(\ln \frac{u_t}{u_y} \right)_x + \left(\ln \frac{u_x}{u_t} \right)_y + \left(\ln \frac{u_y}{u_x} \right)_t = 0,$$

$$\alpha e^{u_{xy}} + \beta e^{u_{xt}} + \gamma e^{u_{yt}} = 0,$$

$$\alpha e^{u_{xx}} + \beta e^{u_{yy}} + \gamma e^{u_{tt}} = 0,$$

respectively. In different context, these and similar dispersionless equations appeared in [4, 5, 6, 13], see also references therein. One can show that all of the above difference equations inherit hydrodynamic reductions of their dispersionless limits, at least to the order ϵ^4 . This property can be used to classify integrable difference equations within particularly interesting subclasses. As a simple illustration let us consider equations of the form

$$\Delta_x^+ \Delta_x^-(u) + \Delta_y^+ \Delta_y^-(f(u)) + \Delta_t^+ \Delta_t^-(g(u)) = 0,$$

where f and g are functions to be determined. The requirement that hydrodynamic reductions of its dispersionless limit, $u_{xx} + f(u)_{yy} + g(u)_{tt} = 0$, can be deformed into reductions of the difference equation up to the order ϵ^2 , leads to the following constraints for f and g :

$$f'' + g'' = 0, \quad g''(1 + f') - g'f'' = 0, \quad f''^2(1 + 2f') - f'(f' + 1)f''' = 0.$$

Without any loss of generality this gives $f(u) = \ln(1 - e^u) - u$, $g(u) = -\ln(1 - e^u)$, resulting in the difference equation

$$\Delta_x^+ \Delta_x^-(u) + \Delta_y^+ \Delta_y^-(\ln(1 - e^u) - u) - \Delta_t^+ \Delta_t^-(\ln(1 - e^u)) = 0,$$

which is yet another equivalent form of the Hirota equation, known as the ‘gauge-invariant form’ [32]. Its dispersionless limit appeared recently in the classification of integrable equations possessing the ‘central quadric ansatz’ [12]. We hope to report further classification results elsewhere.

2 Classification scheme

Let us illustrate our approach using the example of the 2+1 dimensional Toda equation,

$$u_t = u \Delta_y^- w, \quad w_x = \Delta_y^+ u.$$

Expanding the right hand sides using Taylor’s formula one obtains

$$\begin{aligned} \frac{u_t}{u} &= w_y - \frac{\epsilon}{2} w_{yy} + \frac{\epsilon^2}{6} w_{yyy} + \dots, \\ w_x &= u_y + \frac{\epsilon}{2} u_{yy} + \frac{\epsilon^2}{6} u_{yyy} + \dots \end{aligned} \tag{3}$$

The corresponding dispersionless limit results upon setting $\epsilon = 0$:

$$u_t = u w_y, \quad w_x = u_y. \tag{4}$$

This dispersionless system admits exact solutions of the form

$$u = R, \quad w = w(R), \tag{5}$$

where $R(x, y, t)$ satisfies the pair of Hopf-type equations,

$$R_y = \mu R_x, \quad R_t = \mu^2 R R_x. \tag{6}$$

Here $\mu(R)$ is an arbitrary function, and $w' = \mu$. Solutions of this type are known as one-phase solutions (or planar simple waves, or one-component hydrodynamic reductions). One can show that both relations (5), (6) can be deformed into solutions of the full Toda equation in the form

$$u = R, \quad w = w(R) + \epsilon w_1 R_x + \epsilon^2 (w_2 R_{xx} + w_3 R_x^2) + O(\epsilon^3), \tag{7}$$

and

$$\begin{aligned} R_y &= \mu R_x + \epsilon^2 (\alpha_1 R_{xxx} + \alpha_2 R_x R_{xx} + \alpha_3 R_x^3) + O(\epsilon^4), \\ R_t &= \mu^2 R R_x + \epsilon^2 (\beta_1 R_{xxx} + \beta_2 R_x R_{xx} + \beta_3 R_x^3) + O(\epsilon^4), \end{aligned} \tag{8}$$

where w_i, α_i, β_i are certain functions of R . We point out that, modulo the Miura group [8], the relation $u = R$ can be assumed undeformed. Furthermore, one can show that odd order ϵ -corrections in the equations (8) (but not (7)) must vanish identically. Substituting (7) into (3), using (8) and the compatibility condition $R_{yt} = R_{ty}$, one can explicitly calculate all coefficients in (7) and (8) in terms of μ and its derivatives by collecting terms at different powers of ϵ [10].

This gives

$$\begin{aligned}
w_1 &= \frac{1}{2}\mu^2, \\
w_2 &= \frac{1}{12}\mu^2 (2\mu + R\mu'), \\
w_3 &= \frac{1}{24} \left(R(\mu')^2 (2\mu - R\mu') + \mu^2 (11\mu' + R\mu'') \right), \\
\alpha_1 &= \frac{1}{12}R\mu^2\mu', \\
\alpha_2 &= \frac{1}{12}R \left((\mu')^2 (4\mu - R\mu') + 2\mu^2\mu'' \right), \\
\alpha_3 &= \frac{1}{24}R \left(3\mu'\mu'' (2\mu - R\mu') + \mu^2\mu^{(3)} \right), \\
\beta_1 &= \frac{1}{12}R\mu^3 (\mu + 2R\mu'), \\
\beta_2 &= \frac{1}{12}R\mu \left(R(\mu')^2 (11\mu - 2R\mu') + 4\mu^2 (3\mu' + R\mu'') \right), \\
\beta_3 &= \frac{1}{12}R \left(R(\mu')^3 (2\mu - R\mu') + 8R\mu^2\mu'\mu'' + \mu(\mu')^2 (11\mu - 3R^2\mu'') + \mu^3 (4\mu'' + R\mu^{(3)}) \right),
\end{aligned}$$

etc. We point out that this calculation is an entirely algebraic procedure. Thus, all one-phase solutions of the dispersionless system are ‘inherited’ by the original dispersive equation, at least to the order $O(\epsilon^4)$: it is still an open problem to prove their inheritance to all orders in the deformation parameter ϵ . It is important that this works for *arbitrary* $\mu(R)$. The requirement of the inheritance of all hydrodynamic reductions of the dispersionless limit by the full dispersive equation is very restrictive (even to the order $O(\epsilon^2)$), and can be used as an efficient classification criterion in the search for integrable equations. In all examples considered so far, the existence of such deformations to the order ϵ^4 was already sufficient for integrability (in many cases, even the order ϵ^2 was enough), and implied the existence of conventional Lax pairs.

As an illustration, let us consider the class of Toda-type equations of the form

$$u_t = f\Delta_y^- g, \quad w_x = \Delta_y^+ u$$

where $f(u, w)$ and $g(u, w)$ are two arbitrary functions. Imposing the requirement that all one-phase solutions of the corresponding dispersionless limit are inherited by the full dispersive equation, we obtain very strong constraints for f and g . Indeed at the order ϵ we get

$$g_u = 0, \quad f_u f_w = 0, \quad f_w (f g_{ww} + g_w f_w) = 0,$$

so that, excluding the degenerate case $f_u = 0$, we arrive at $g_u = 0$, $f_w = 0$. At the order ϵ^2 one obtains two additional constraints:

$$f''(u) = 0, \quad g''(w)^2 - g'(w)g'''(w) = 0.$$

Modulo elementary changes of variables this leads to the cases $f(u) = u$, $g(w) = w$ and $f(u) = \alpha u + \beta$, $g(w) = e^w$ which correspond to the Toda and Volterra chains, respectively (see Section 3.2). Note that all these constraints appear at the order ϵ^2 , and are already sufficient for the integrability, implying the existence of Lax pairs.

Remark. All equations discussed in this paper possess dispersionless limits of the form

$$u_t = \varphi u_x + \psi u_y + \eta w_y, \quad w_x = u_y, \quad (9)$$

which will be assumed *non-degenerate* in the following sense:

- (i) The coefficient η is nonzero: this is equivalent to the requirement that the corresponding dispersion relation defines an irreducible conic.
- (ii) The dispersionless limit (9) is not *totally linearly degenerate*. Recall that totally linearly degenerate systems are characterised by the relations [11]

$$\eta_w = 0, \quad \psi_w + \eta_u = 0, \quad \varphi_w + \psi_u = 0, \quad \varphi_u = 0.$$

Dispersive deformations of degenerate systems do not inherit hydrodynamic reductions, and require a different approach which is beyond the scope of this paper. We point out that most of the integrable examples of interest are non-degenerate in the above sense, or can be brought into a non-degenerate form.

3 Differential-difference equations

In this section we search for integrable examples within various classes of differential-difference PDEs generalising intermediate long wave and Toda type equations. We skip the details of calculations, which essentially follow the pattern outlined in Sect. 2. Classification results are presented modulo Galilean transformations, and transformations of the form $u \rightarrow \alpha u + \beta$, $w \rightarrow \alpha w + \gamma$.

3.1 Equations with the intermediate long wave non-locality: $\Delta_x^+ w = \frac{T_x+1}{2} u_y$

First we classify integrable equations of the form

$$u_t = \varphi u_x + \psi u_y + \tau w_x + \eta w_y + \epsilon(\dots) + \epsilon^2(\dots), \quad (10)$$

where w is the non-locality of the intermediate long wave type, $\Delta_x^+ w = \frac{T_x+1}{2} u_y$, or, equivalently, $w = \frac{\epsilon}{2} \frac{T_x+1}{T_x-1} u_y$. Here dots denote terms which are homogeneous polynomials of degree two and three in the x - and y -derivatives of u and w , whose coefficients are allowed to be functions of u and w . One can show that all ϵ -terms, as well as all terms containing derivatives with respect to x , in particular φ and τ , must vanish identically.

Theorem 2 *The following examples constitute a complete list of integrable equations of the form (10) with the non-locality of intermediate long wave type:*

$$u_t = uu_y + w_y, \quad (11)$$

$$u_t = (w + \alpha e^u) u_y + w_y, \quad (12)$$

$$u_t = u^2 u_y + (uw)_y + \frac{\epsilon^2}{12} u_{yyy}, \quad (13)$$

$$u_t = u^2 u_y + (uw)_y + \frac{\epsilon^2}{12} \left(u_{yy} - \frac{3}{4} \frac{u_y^2}{u} \right)_y. \quad (14)$$

Although the first two equations are of the first order, they should be viewed as dispersive: the dispersion is contained in the equation for nonlocality. The equation (11), which can be written in the form

$$u_t = uu_y + \frac{\epsilon}{2} \frac{T_x + 1}{T_x - 1} u_{yy},$$

first appeared in [7] as a differential-difference analogue of the KP equation, see also [27] (we point out that its dispersionless limit does not coincide with that of KP). It can also be viewed as a 2+1 dimensional integrable version of the intermediate long wave equation [35]. The equation (13) is a differential-difference version of the Veselov-Novikov equation discussed in [25]. The last example can be viewed as a differential-difference version of the modified Veselov-Novikov equation. To the best of our knowledge, equations (12) and (14) are new.

Lax pairs, dispersionless limits and dispersionless Lax pairs for the equations from Theorem 2 are provided in the table below (note that equations (13) and (14) have coinciding dispersionless limits/dispersionless Lax pairs). Here and in what follows, Lax pairs were obtained by the quantisation of dispersionless Lax pairs as discussed in [34].

<i>Eqn</i>	<i>Lax pair</i>	<i>Dispersionless limit</i>	<i>Dispersionless Lax pair</i>
(11)	$T_x \psi = \epsilon \psi_y - u \psi$ $\epsilon \psi_t = \frac{\epsilon^2}{2} \psi_{yy} + (w - \frac{\epsilon}{2} u_y) \psi$	$u_t = uu_y + w_y$ $w_x = u_y$	$e^{S_x} = S_y - u$ $S_t = \frac{1}{2} S_y^2 + w$
(12)	$T_x \psi = \epsilon e^{-u} \psi_y - \alpha \psi$ $\psi_t = \frac{\epsilon}{2} \psi_{yy} + (w - \frac{\epsilon}{2} u_y) \psi_y$	$u_t = (w + \alpha e^u) u_y + w_y$ $w_x = u_y$	$e^{S_x} = e^{-u} S_y - \alpha$ $S_t = \frac{1}{2} S_y^2 + w S_y$
(13)	$\epsilon(T_x - 1) \psi_y = -2u(T_x + 1) \psi$ $\psi_t = \frac{\epsilon^2}{12} \psi_{yyy} + (w - \frac{\epsilon}{2} u_y) \psi_y$	$u_t = u^2 u_y + (uw)_y$ $w_x = u_y$	$\frac{e^{S_x} - 1}{e^{S_x} + 1} S_y = -2u$ $S_t = \frac{1}{12} S_y^3 + w S_y$
(14)	$\epsilon(T_x - 1) \psi_y = \frac{\epsilon}{2} \frac{u_y}{u} (T_x - 1) \psi - 2u(T_x + 1) \psi$ $\psi_t = \frac{\epsilon^2}{12} \psi_{yyy} + (w - \frac{\epsilon}{2} u_y) \psi_y + \frac{1}{2} (w_y - \frac{\epsilon}{2} u_{yy}) \psi$	$u_t = u^2 u_y + (uw)_y$ $w_x = u_y$	$\frac{e^{S_x} - 1}{e^{S_x} + 1} S_y = -2u$ $S_t = \frac{1}{12} S_y^3 + w S_y$

Remark. Equations (11) and (12) are related by a (rather non-trivial) gauge transformation. Let us begin with the dispersionless limit of (12),

$$u_t = (w + \alpha e^u) u_y + w_y, \quad w_x = u_y,$$

with the corresponding Lax pair

$$S_t = \frac{1}{2} S_y^2 + w S_y, \quad e^{S_x} = e^{-u} S_y - \alpha.$$

Let h be a potential such that $u = h_x$, $w = h_y$. One can verify that the new variables $\tilde{u} = w + \alpha e^u$, $\tilde{w} = h_t - \frac{w^2}{2}$, $\tilde{S} = S + h$ satisfy the dispersionless equation (11),

$$\tilde{u}_t = \tilde{u} \tilde{u}_y + \tilde{w}_y, \quad \tilde{w}_x = \tilde{u}_y,$$

along with the corresponding Lax pair

$$\tilde{S}_t = \frac{1}{2}\tilde{S}_y^2 + \tilde{w}, \quad e^{\tilde{S}_x} = \tilde{S}_y - \tilde{u},$$

thus establishing the required link at the dispersionless level (it is sufficient to perform this calculation at the level of Lax pairs: the equations for \tilde{u}, \tilde{w} will be automatic). The dispersive version of this construction is as follows. We take the equation (12),

$$u_t = (w + \alpha e^u)u_y + w_y, \quad w = \frac{\epsilon}{2} \frac{T_x + 1}{T_x - 1} u_y,$$

with the corresponding Lax pair

$$\psi_t = \frac{\epsilon}{2}\psi_{yy} + (w - \frac{\epsilon}{2}u_y)\psi_y, \quad T_x\psi = \epsilon e^{-u}\psi_y - \alpha\psi.$$

Let H be a potential such that $u = \frac{T_x - 1}{\epsilon}H$, $w = \frac{T_x + 1}{2}H_y$. One can verify that the new variables $\tilde{u} = H_y + \alpha e^u$, $\tilde{w} = H_t - \frac{H_y^2}{2} + \frac{\alpha\epsilon}{2}e^u\Delta_x^+ H_y$, $\tilde{\psi} = e^{H/\epsilon}\psi$ satisfy the equation (11),

$$\tilde{u}_t = \tilde{u}\tilde{u}_y + \tilde{w}_y, \quad \tilde{w} = \frac{\epsilon}{2} \frac{T_x + 1}{T_x - 1} \tilde{u}_y,$$

with the corresponding Lax pair

$$\epsilon\tilde{\psi}_t = \frac{\epsilon^2}{2}\tilde{\psi}_{yy} + (\tilde{w} - \frac{\epsilon}{2}\tilde{u}_y)\tilde{\psi}, \quad T_x\tilde{\psi} = \epsilon\tilde{\psi}_y - \tilde{u}\tilde{\psi}.$$

Again, it is sufficient to perform this calculation at the level of Lax pairs. Due to the complexity of this transformation we prefer to keep both equations in the list of Theorem 2 as separate cases.

Another interesting class of equations with the non-locality of intermediate long wave type is

$$u_t = \psi u_y + \eta w_y + f\Delta_x^+ g + p\Delta_x^- q, \tag{15}$$

where $\Delta_x^+ w = \frac{T_x + 1}{2}u_y$, and ψ, η, f, g, p, q are functions of u and w .

Theorem 3 *The following examples constitute a complete list of integrable equations of the form (15) with the non-locality of intermediate long wave type:*

$$\begin{aligned} u_t &= uu_y + w_y, \\ u_t &= (w + \alpha e^u)u_y + w_y, \\ u_t &= wu_y + w_y + \frac{\Delta_x^+ + \Delta_x^-}{2}e^{2u}, \end{aligned} \tag{16}$$

$$u_t = wu_y + w_y + e^u(\Delta_x^+ + \Delta_x^-)e^u. \tag{17}$$

Here the first two equations are the same as in Theorem 2, the third example first appeared in [18], while the fourth is apparently new. Lax pairs, dispersionless limits and dispersionless Lax pairs for equations from Theorem 3 are provided in the table below (note that equations (16) and (17) have coinciding dispersionless limits):

<i>Eqn</i>	<i>Lax pair</i>	<i>Dispersionless limit</i>	<i>Dispersionless Lax pair</i>
(16)	$\begin{aligned}\epsilon\psi_y &= (T_x e^u)T_x\psi + e^u T_x^{-1}\psi \\ \epsilon\psi_t &= \frac{1}{2}e^{T_x(1+T_x)u}T_x^2\psi - \frac{1}{2}e^{(1+T_x^{-1})u}T_x^{-2}\psi + \\ &\quad T_x(we^u)T_x\psi + we^u T_x^{-1}\psi\end{aligned}$	$\begin{aligned}u_t &= 2e^{2u}u_x + wu_y + w_y \\ w_x &= u_y\end{aligned}$	$\begin{aligned}S_y &= 2e^u \cosh S_x \\ S_t &= e^{2u} \sinh 2S_x + \\ &\quad 2we^u \cosh S_x\end{aligned}$
(17)	$\begin{aligned}\epsilon\psi_y &= e^u(T_x\psi + T_x^{-1}\psi) \\ \epsilon\psi_t &= \frac{1}{2}e^{(1+T_x)u}T_x^2\psi - \frac{1}{2}e^{(1+T_x^{-1})u}T_x^{-2}\psi + \\ &\quad we^u(T_x\psi + T_x^{-1}\psi) + \frac{\epsilon}{2}e^u[(\Delta_x^+ + \Delta_x^-)e^u]\psi\end{aligned}$	$\begin{aligned}u_t &= 2e^{2u}u_x + wu_y + w_y \\ w_x &= u_y\end{aligned}$	$\begin{aligned}S_y &= 2e^u \cosh S_x \\ S_t &= e^{2u} \sinh 2S_x + \\ &\quad 2we^u \cosh S_x\end{aligned}$

3.2 Equations with the Toda non-locality: $w_x = \Delta_y^+ u$

In this section we classify integrable equations of the form

$$u_t = \varphi u_x + f \Delta_y^+ g + p \Delta_y^- q, \quad (18)$$

where the non-locality w is defined as $w_x = \Delta_y^+ u$, and φ, f, g, p, q are functions of u and w .

Theorem 4 *The following examples constitute a complete list of integrable equations of the form (18) with the non-locality of Toda type:*

$$u_t = u \Delta_y^- w, \quad (19)$$

$$u_t = (\alpha u + \beta) \Delta_y^- e^w, \quad (20)$$

$$u_t = e^w \sqrt{u} \Delta_y^+ \sqrt{u} + \sqrt{u} \Delta_y^- (e^w \sqrt{u}), \quad (21)$$

here $\alpha, \beta = \text{const.}$

Equation (19) is the 2+1 dimensional Toda equation, which can also be written in the form $(\ln u)_{xt} = \Delta_y^+ \Delta_y^- u$, while equation (20) is equivalent to the Volterra chain when $\alpha \neq 0$, or to the Toda chain when $\alpha = 0$. Lax pairs, dispersionless limits and dispersionless Lax pairs for the equations from Theorem 4 are provided in the table below:

<i>Eqn</i>	<i>Lax pair</i>	<i>Dispersionless limit</i>	<i>Dispersionless Lax pair</i>
(19)	$\epsilon T_y \psi_x = u \psi$ $\epsilon \psi_t = -T_y \psi + (T_y^{-1} w) \psi$	$u_t = u w_y$ $w_x = u_y$	$e^{S_y} S_x = u$ $S_t = -e^{S_y} + w$
(20)	$\epsilon T_y \psi_x = -(T_y u) T_y \psi + (\alpha T_y u + \beta) \psi$ $\epsilon \psi_t = -e^w T_y \psi + \alpha e^w \psi$	$u_t = (\alpha u + \beta) e^w w_y$ $w_x = u_y$	$e^{S_y} S_x = -u e^{S_y} + \alpha u + \beta$ $S_t = -e^w e^{S_y} + \alpha e^w$
(21)	$\epsilon T_y \psi_x = \epsilon \sqrt{\frac{T_y u}{u}} \psi_x - (T_y u) T_y \psi - \sqrt{u T_y u} \psi$ $\epsilon \psi_t = \frac{1}{2} e^w T_y \psi - \frac{1}{2} (T_y^{-1} e^w) T_y^{-1} \psi$	$u_t = e^w u_y + u e^w w_y$ $w_x = u_y$	$e^{S_y} S_x = S_x - u e^{S_y} - u$ $S_t = e^w \sinh S_y$

Remark. One can show that there exist no non-degenerate integrable equations of the form

$$u_t = \eta w_y + f \Delta_x^+ g + p \Delta_x^- q,$$

where the non-locality w is defined as $\Delta_x^+ w = u_y$, and η, f, g, p, q are functions of u and w . Indeed, the integrability requirement implies the condition $\eta = 0$, which corresponds to degenerate systems.

3.3 Equations with the fully discrete non-locality: $\Delta_x^+ w = \Delta_y^+ u$

In this last section we classify integrable equations of the form

$$u_t = f \Delta_x^+ g + h \Delta_x^- k + p \Delta_y^+ q + r \Delta_y^- s, \quad (22)$$

where the non-locality w is defined as $\Delta_x^+ w = \Delta_y^+ u$, and the functions f, g, h, k, p, q, r, s depend on u and w .

Theorem 5 *The following examples constitute a complete list of integrable equations of the form (22) with the fully discrete non-locality:*

$$u_t = u \Delta_y^-(u - w), \quad (23)$$

$$u_t = u(\Delta_x^+ + \Delta_y^-)w, \quad (24)$$

$$u_t = (\alpha e^{-u} + \beta) \Delta_y^- e^{u-w}, \quad (25)$$

$$u_t = (\alpha e^u + \beta)(\Delta_x^+ + \Delta_y^-)e^w, \quad (26)$$

$$u_t = \sqrt{\alpha - \beta e^{2u}} \left(e^{w-u} \Delta_y^+ \sqrt{\alpha - \beta e^{2u}} + \Delta_y^-(e^{w-u} \sqrt{\alpha - \beta e^{2u}}) \right), \quad (27)$$

here $\alpha, \beta = \text{const.}$

In equivalent form, equation (27) is known as the $2 + 1$ dimensional analogue of the modified Volterra lattice [30]. Lax pairs, dispersionless limits and dispersionless Lax pairs for the equations from Theorem 5 are provided in the table below:

<i>Eqn</i>	<i>Lax pair</i>	<i>Dispersionless limit</i>	<i>Dispersionless Lax pair</i>
(23)	$T_x T_y \psi = -T_y \psi + (T_y u) T_x \psi$ $\epsilon \psi_t = T_y \psi - w \psi$	$u_t = u(u_y - w_y)$ $w_x = u_y$	$e^{S_x + S_y} = -e^{S_y} + u e^{S_x}$ $S_t = e^{S_y} - w$
(24)	$T_x T_y \psi = T_y \psi - w \psi$ $\epsilon \psi_t = T_y \psi + (T_y^{-1} w) \psi$	$u_t = u(u_y + w_y)$ $w_x = u_y$	$e^{S_x + S_y} = e^{S_y} - u$ $S_t = e^{S_y} + w$
(25)	$T_y^{-1} \psi = \frac{e^u}{\alpha + \beta e^u} T_x^{-1} \psi + \frac{1}{\alpha + \beta e^u} \psi$ $\epsilon T_x^{-1} \psi_t = -\epsilon e^{-u} \psi_t - \alpha e^{-w} T_x^{-1} \psi + \beta e^{-w} \psi$	$u_t = (\alpha + \beta e^u) e^{-w} (u_y - w_y)$ $w_x = u_y$	$e^{-S_y} = \frac{e^u e^{-S_x} + 1}{\alpha + \beta e^u}$ $e^{-S_x} S_t = -e^{-u} S_t - \alpha e^{-w} e^{-S_x} + \beta e^{-w}$
(26)	$T_y^{-1} \psi = -\frac{e^u}{\alpha e^u + \beta} T_x \psi + \frac{1}{\alpha e^u + \beta} \psi$ $\epsilon T_x \psi_t = \epsilon e^{-u} \psi_t - \beta (T_x e^w) T_x \psi - \alpha (T_x e^w) \psi$	$u_t = (\alpha e^u + \beta) e^w (u_y + w_y)$ $w_x = u_y$	$e^{-S_y} = \frac{-e^u e^{S_x} + 1}{\alpha e^u + \beta}$ $e^{S_x} S_t = -e^{-u} S_t - \beta e^w e^{S_x} - \alpha e^w$
(27)	$T_x T_y \psi = \frac{\alpha}{\beta} (T_y e^{-u}) T_y \psi + \frac{T_y (e^{-u} \sqrt{\alpha - \beta e^{2u}})}{\sqrt{\alpha - \beta e^{2u}}} (T_x \psi - e^u \psi)$ $\epsilon \psi_t = -\alpha e^w T_y \psi + \beta (T_y^{-1} e^w) T_y^{-1} \psi$	$u_t = \alpha (e^{w-u})_y - \beta (e^{w+u})_y$ $w_x = u_y$	$e^{S_x + S_y} = \frac{\alpha}{\beta} e^{-u} e^{S_y} + e^{-u} e^{S_x} - 1$ $S_t = -\alpha e^w e^{S_y} + \beta e^w e^{-S_y}$

Remark 1. The continuum limit of the modified Volterra lattice (27) in x -direction, namely $x \rightarrow hx$, $u \rightarrow hu$ and $h \rightarrow 0$, gives the Toda-type lattice (21). Similarly, in the same limit equations (23) and (24) give the Toda equation (19), while the remaining two, (25) and (26), lead to the equation (20) with $\alpha = 0$.

Remark 2. We point out that there exist other types of integrable equations with the non-locality $\Delta_x^+ w = \Delta_y^+ u$, which are not covered by Theorem 5. One of such examples is the first flow of the discrete modified Veselov-Novikov hierarchy constructed in [33],

$$u_t = \sqrt{-(T_y^{-1} \Delta_x^+ e^{2w})(\Delta_x^+ e^{-2w})}, \quad \Delta_x^+ w = \Delta_y^+ u.$$

This equation is not of the form (22), furthermore, its dispersionless limit is degenerate:

$$u_t = 2w_x, \quad w_x = u_y.$$

4 Concluding remarks

This paper outlines an approach to the classification of integrable differential-difference equations in 2+1D (with one/two discrete variables) based on the method of hydrodynamic reductions.

1. It would be challenging to extend our approach to the classification of fully discrete 3D equations. Some work in this direction, based on the concept of multidimensional consistency, has been done in [2, 29].
2. A novel approach to the integrability of multidimensional lattices of the Toda and Volterra type, based on the concept of characteristic Lie rings, was proposed recently in [14]. Its relation to our method is yet to be properly investigated.
3. Our calculations demonstrate that, even though nontrivial combinations of differential and difference operators (in the same independent variable) were allowed in the initial ansatz, non of such ‘differential-delay’ cases survived the integrability test, leading to either purely discrete, or purely differential situations. It remains to be seen whether there exist proper differential-delay integrable equations in 2+1 D.

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